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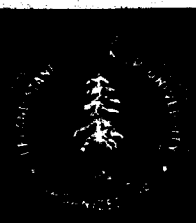

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# A METHOD OF DISCUSSION OF THE ELASTIC STABILITY IN NON-LINEAR SNAP BUCKLING

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# A METHOD OF DISCUSSION OF THE ELASTIC STABILITY IN NON-LINEAR SNAP BUCKLING

## (SNAP BUCKLING OF CIRCULAR PLATES DUE TO RAPID HEATING OR COOLING)

by

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ABSTRACT

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In order to investigate the stability condition of certain equilibrium states existing in the non-linear buckling phenomenon as a result of finite deformations, and in order to specify a reasonable deformation process, a method of analysis of the stability criterion in two-dimensional plates or shells is presented. This method is applied to a sample problem concerning the snap buckling of circular plates due to rapid heating or cooling and is found to be of use especially when the Euler equation derived from the variational method is solved by the progressive approximation method. Since the effect of the temperature gradient through the thickness resulting from the rapid heating or cooling corresponds to the action of a pressure over the surface, it should be noted that a snap buckling may be induced by this effect, even if there is no initial deflection.

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## NOMENCLATURE

| | |
|---|---|
| $a$ | radius of a circular plate |
| $C_p$ | specific heat of plate |
| $d$ | thickness of a circular plate |
| $E$ | Young's modulus |
| $e_1, e_2$ | radial and circumferential strains respectively |
| $F$ | free energy in the total volume of a circular plate |
| $F'$ | free energy per unit volume |
| $f(t), g(t)$ | time functions representing the variations of $\overline{T}$ and $\tilde{T}$ |
| $h$ | heat transfer coefficient |
| $k$ | heat conductivity of plate |
| $p_n$ | eigenvalue defined by Eq.(34) |
| $q$ | heating rate |
| $r$ | axial coordinate on median surface of circular plate |
| $T$ | temperature rise above the initial unstrained state |
| $T_{aw}$ | adiabatic wall temperature |
| $T_o$ | a constant temperature |
| $T_w$ | upper surface temperature |
| $t$ | time after heating |
| $t_1$ | time after cooling |
| $u, w$ | radial and normal displacements; $w$ denotes the normal deflection from the flate state including the initial deflection |
| $w_o$ | initial deflection |
| $x$ | $\alpha T_o(a^2/d^2)$ ; nondimensional temperature rise |

NOMENCLATURE (Cont'd)

| | |
|---|---|
| $z$ | normal coordinate in the direction of thickness in a circular plate |
| $\alpha$ | linear coefficient of thermal expansion |
| $\delta$ | largest deflection at the center of plate in the heated state |
| $\delta_o$ | initial largest deflection at the center of plate |
| $\zeta$ | $r/a$ ; nondimensional radial coordinate |
| $\theta$ | circumferential coordinate on median surface of circular plate |
| $\kappa_1, \kappa_2$ | radial and circumferential changes of curvature, respectively |
| $\nu$ | Poisson's ratio |
| $\pi$ | total potential energy |
| $\rho$ | density of plate |
| $\sigma_1, \sigma_2$ | radial and circumferential stresses, respectively |

Subscripts:

| | |
|---|---|
| $r$ and $\zeta$ | denote the differentiation with respect to $r$ and $\zeta$ , respectively |
| $o$ | denotes the components concerning the initial state |
| $\bar{(\ )}$ | denotes the components on the median surface |

## 1. INTRODUCTION

There have been few papers which discussed the stability criterion of certain equilibrium states existing in the nonlinear buckling phenomenon, except in the one-dimensional case. It is convenient to rely on energy considerations in order to discuss the stability criterion in elastic problems, treating them as variational problems. However, in nonlinear problems, it is generally so difficult to solve the Euler equation derived from the variational process that such direct methods as the Ritz Method and the Galerkin Method have usually been employed. It is true that these methods are sometimes useful to get appropriate solutions easily and then to discuss the stability criterion, even if some deflection modes are fixed, as seen in the one-dimensional case of bars treated by Marguerre[1*]. However, the direct methods in the two-dimensional cases, such as plates and shells, are generally of doubtful accuracy, as shown in a previous paper[2] and the introduction of terms with unknown coefficients in the expression of the deflection mode in order to improve the accuracy makes the mathematical analysis more complicated. In view of the above considerations, it was shown in a previous paper[3] that a more exact solution could be obtained easily by applying a progressive-approximation method to the Euler equation. In such cases, the stability criterion can be analyzed by the "extremum" discussion in variational mathematics; however, it is very difficult, especially in two-dimensional nonlinear problems, to find the necessary and sufficient conditions for it.

Accordingly, in this paper, the logic of a method of discussing the stability criterion in elastic energy problems is presented and the method is applied to a sample problem concerning the snap buckling of circular plates in order to verify its appropriateness.

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* Superscript numbers indicate the References collected at the end of the paper.

As discussed recently in previous papers concerning the thermal deformation of various structural elements due to rapid heating[4,6], the effect of a temperature gradient through the thickness of a plate corresponds to the action of a normal pressure, depending on the temperature gradient distribution over the surface. As a result of this effect, it is shown by means of a progressive-approximation method, that a thermal deformation or thermal snap buckling takes place, even if there is no initial deflection, and a reasonable deformation process is specified on the basis of a discussion of the stability criterion.

## 2. DERIVATION OF EQUILIBRIUM EQUATIONS FOR A CIRCULAR PLATE HAVING INITIAL DEFLECTIONS

A circular plate subjected to aerodynamic heating from the upper surface is considered here, and the existence of a transient temperature gradient through its thickness is taken into account as shown in Figure 1. For the sake of simplicity, the only case considered is that in which the distributions of temperature, deformation and so on, are axisymmetrical. Therefore, the shear strain components vanish in the two-dimensional plane stress state and the stress versus strain relations are expressed by

$$\left.\begin{aligned} Ee_1 &= \sigma_1 - \nu\sigma_2 + \alpha TE \\ Ee_2 &= \sigma_2 - \nu\sigma_1 + \alpha TE \end{aligned}\right\} \tag{1}$$

If it is assumed that, in comparatively thin plates, a normal plane across the plane remains a plane after deformation, the strain components $e_1$ and $e_2$ can be given by

$$\left.\begin{aligned} e_1 &= \bar{e}_1 - z\kappa_1 \,, \\ e_2 &= \bar{e}_2 - z\kappa_2 \end{aligned}\right\} \tag{2}$$

If large deformations are taken into account, and only terms of the first order of magnitude are retained, the strains and changes of curvature can be given by the displacements and their derivatives, as follows:

$$\left.\begin{aligned} \bar{e}_1 &= u_r + (w_r^2 - w_{or}^2)/2 \\ \bar{e}_2 &= u/r \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} \kappa_1 &= w_{rr} - w_{orr} \\ \kappa_2 &= (w_r - w_{or})/r \end{aligned}\right\} \tag{4}$$

According to the stationary principle of the total potential energy $\pi$ , the equilibrium equations will be derived through the variational process. For the case of isothermal thermoelasticity, the "free energy" $F'$ replaces the usual strain energy and $F'$ is expressed by

$$F' = \frac{E}{2(1-\nu^2)} (e_1^2+2\nu e_1 e_2+e_2^2) - \frac{\alpha T E}{1-\nu} (e_1+e_2) + C_1(T) \tag{5}$$

A circular plate is considered to be subjected to an aerodynamic pressure $q(r)$ in addition to an aerodynamic heating. Consequently, $\pi$ is shown as

$$\begin{aligned}\pi &= \int_V F'dV + 2\pi \int_o^a q(r)(w-w_o)rdr \\ &= \frac{\pi E d}{1-\nu^2}\left[\int_o^a (\bar{e}_1^2+2\nu\bar{e}_1\bar{e}_2+\bar{e}_2^2)rdr + \frac{d^2}{12}\int_o^a (\kappa_1^2+2\nu\kappa_1\kappa_2+\kappa_2^2)rdr\right] \\ &- \frac{2\pi\alpha E}{1-\nu}\left[\int_o^a (\bar{e}_1+\bar{e}_2)\bar{T}rdr - \int_o^a (\kappa_1+\kappa_2)\tilde{T}rdr\right] \\ &+ 2\pi d\int_o^a C_1(T)rdr + 2\pi\int_o^a q(w-w_o)rdr \end{aligned} \tag{6}$$

where $$\bar{T} = \int_{d/2}^{d/2} Tdz \quad , \quad \tilde{T} = \int_{d/2}^{d/2} \bar{T}zdz$$

After introduction of Eqs.(3) and (4) into Eq.(5), the variations with respect to the displacements $u$ and $w$ lead to the following equilibrium equations based on the stationary condition $\delta\pi = 0$ :

In the r-direction:

$$\frac{Ed}{1-\nu^2}\left[\frac{d}{dr}\left\{r\left[u_r+\frac{1}{2}(w_r^2-w_{or}^2)\right]\right\} - \frac{\nu}{2}(w_r^2-w_{or}^2) - \frac{u}{r}\right] = \frac{\alpha E}{1-\nu}\bar{T}_r \cdot r \tag{7a}$$

In the z-direction:

$$\frac{Ed^3}{12(1-\nu^2)}\left[\frac{d^2}{dr^2}\left\{r(w_{rr}-w_{orr})\right\} - \frac{d}{dr}\left\{\frac{1}{r}(w_r-w_{or})\right\}\right]$$
$$- \frac{Ed}{1-\nu^2}\left[\frac{d}{dr}\left\{w_r\cdot r\left[u_r+\frac{1}{2}(w_r^2-w_{or}^2)\right]\right\} + \nu\frac{d}{dr}(w_r\cdot u)\right]$$
$$+ \frac{\alpha E}{1-\nu}\left[\frac{d}{dr}(w_r\cdot\bar{T}\cdot r) + \frac{d}{dr}(\tilde{T}_r\cdot r)\right] + qr = 0 \qquad (7b)$$

If the stress components on the median surface $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are introduced by the following relations

$$E\bar{e}_1 = \bar{\sigma}_1 - \nu\bar{\sigma}_2 + \frac{\alpha E}{d}\bar{T}\ , \qquad E\bar{e}_2 = \bar{\sigma}_2 - \nu\bar{\sigma}_1 + \frac{\alpha E}{d}\bar{T} \qquad (8)$$

and then the stress function is used that is defined by

$$\bar{\sigma}_1 = \phi/r\ , \qquad \bar{\sigma}_2 = d\phi/dr \qquad (9)$$

which satisfies the equilibrium equation (7a) in the radial direction; the equilibrium equation (7b) in the normal direction is finally reduced to:

$$\frac{Ed^3}{12(1-\nu^2)}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left[(w_r-w_{or})r\right]\right\}\right] = d\cdot\frac{d}{dr}(w_r\phi) - \frac{\alpha E}{1-\nu}\frac{d}{dr}\left(\frac{d\tilde{T}}{dr}\cdot r\right) - qr \qquad (10)$$

If u is omitted from both of Eqs.(3) and Eqs.(8) and (9) are used, the compatibility equation can be obtained as follows:

$$r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(\phi r)\right\} = -\frac{\alpha E}{d}r\frac{d\bar{T}}{dr} - \frac{E}{2}(w_r^2-w_{or}^2) \qquad (11)$$

The natural boundary conditions, which are given as the by-product of the variational procedure, are as follows at $r = 0\ ,\ a$ :

$$
\left.
\begin{aligned}
&\delta u = 0 \quad \text{or} \quad u_r + \frac{1}{2}(w_r^2 - w_{or}^2) + \nu \frac{u}{r} - (1+\nu)\frac{\alpha}{d}\bar{T} = 0 \\
&\delta w = 0 \quad \text{or} \quad \frac{Ed}{1-\nu^2} w_r \left\{ u_r + \nu \frac{u}{r} + \frac{1}{2}(w_r^2 - w_{or}^2) \right\} - \frac{Ed^3}{12(1-\nu^2)} \frac{d}{dr}\left[ \frac{1}{r} \frac{d}{dr}\left\{ (w_r - w_{or})r \right\} \right] \\
&\qquad\qquad - \frac{\alpha E}{1-\nu}\left( w_r \bar{T} + \frac{d\tilde{T}}{dr} \right) = 0 \\
&\delta w_r = 0 \quad \text{or} \quad \frac{Ed^3}{12(1-\nu^2)} \left[ (w_{rr} - w_{orr}) + \frac{\nu}{r}(w_r - w_{or}) \right] + \frac{\alpha E}{1-\nu} \tilde{T} = 0
\end{aligned}
\right\} \quad (12)
$$

Finally, this problem is reduced to the simultaneous solution of the two nonlinear differential equations (10) and (11), under the given boundary conditions. It can be easily seen from Eq.(10) that the non-uniform distribution of the temperature gradient through the thickness ($\tilde{T}$) corresponds to the action of the pressure. It should be noted that the second derivatives of $\tilde{T}$ correspond to a normal pressure ($q$) in orthogonal coordinates[5], but in contrast, even the first derivative of $\tilde{T}$ corresponds to $q$ in polar coordinates.

If it is assumed that the temperature distribution is axisymmetrical about the center, $\bar{T}$ and $\tilde{T}$ are expressed as

$$
\frac{\bar{T}}{d} = f(t) \sum_{j=0}^{\infty} T_{2j} \left( \frac{r}{a} \right)^{2j} , \qquad \frac{\tilde{T}}{d^2} = g(t) \sum_{j=0}^{\infty} T_{2j} \left( \frac{r}{a} \right)^{2j} \qquad (13)
$$

where $f(t)$ and $g(t)$ represent the variations given as functions of time.

In the following analytical examples, only the first two terms $j = 0$ and $1$ are retained and $f(t)$ and $g(t)$ are simply expressed as

$$
\bar{T}/d = f(t) T_o (1 - \gamma \zeta^2) , \qquad \tilde{T}/d^2 = g(t) T_o (1 - \gamma \zeta^2) \qquad (14)
$$

where $\zeta = r/a$ .

## 3. EXAMPLES FOR A CIRCULAR PLATE CLAMPED AT THE OUTER EDGE

The basic differential equations (10) and (11) are now rewritten by use of the nondimensional variable $\zeta$ as

$$\zeta \frac{d}{d\zeta}\left\{\frac{1}{\zeta}\frac{d}{d\zeta}(\phi\zeta)\right\} = -\alpha E\left(\frac{a}{d}\right)\zeta\frac{d\tilde{T}}{d\zeta} - \frac{E}{2a}(w_\zeta^2 - w_{o\zeta}^2) \tag{11'}$$

$$\frac{Ed^3}{12(1-\nu^2)}\frac{d}{d\zeta}\left[\zeta\frac{d}{d\zeta}\left\{\frac{1}{\zeta}\frac{d}{d\zeta}\left[(w_\zeta - w_{o\zeta})\zeta\right]\right\}\right] = ad\frac{d}{d\zeta}(w_\zeta\cdot\phi) - \frac{\alpha Ea^2}{1-\nu}\frac{d}{d\zeta}\left(\frac{d\tilde{T}}{d\zeta}\zeta\right) - a^4 q\zeta \tag{10'}$$

In the following analysis, the normal pressure is assumed to be zero, that is, $q = 0$ .

First of all, the initial deflection is considered to be expressed, for example, as

$$w_o = \delta_o(1-\zeta^2)^2 \tag{15}$$

which satisfies the clamped-edge condition. For the cases of other initial deflection modes and temperature distributions, the analytical process to be used in this paper can be followed.

Through adoption of the progressive-approximation method used in a previous paper[3], a deflection mode for $w$ of the first order is assumed to be as follows:

$$w = \delta(1-\zeta^2)^2 \tag{16}$$

which satisfies the boundary condition for the clamped edge. After introducing three of Eqs.(14), (15) and (16) into Eq.(11'), $\phi$ can be obtained by integrating the left-hand side of Eq.(11') successively. The integral constants in $\phi$ can be determined from the following two conditions:

(1) The boundary condition on $u$

$$\zeta = 0,1 \quad ; \quad u = 0 \tag{17}$$

where $u$ is obtained from Eq.(18) which is rewritten by expressing $\bar{e}_1$, $\bar{e}_2$ by $u$, $w$, and $\bar{\sigma}_1$, $\bar{\sigma}_2$ by $\phi$, and

(2) The boundary condition that the stress should be finite at $\zeta = 0$.

Then the stress function $\phi$ and the radial deformation $u$ are expressed by

$$\frac{\phi}{a\zeta} = \frac{E}{6a^2}(\delta^2-\delta_o^2)\left\{\frac{5-3\nu}{1-\nu} - 6\zeta^2 + 4\zeta^4 - \zeta^6\right\}$$
$$+ \frac{E\alpha T_o f(t)}{1-\nu}\left[-1 + \frac{\gamma}{4}\left\{(1+\nu) + (1-\nu)\zeta^2\right\}\right] \tag{18}$$

$$\frac{u}{a} = \frac{1}{6a^2}(\delta^2-\delta_o^2)\left\{(5-3\nu)\zeta - 6(3-\nu)\zeta^3 + 4(5-\nu)\zeta^5 - (7-\nu)\zeta^7\right\}$$
$$+ \alpha T_o f(t)\frac{1+\nu}{4}\gamma(\zeta-\zeta^3) \tag{19}$$

respectively.

The second step is to obtain $w$ of the second approximation by integrating Eq.(16) successively after introducing $\phi$ of Eq.(18) and $w$ of Eq.(16) into the right-hand side of Eq.(10'). The integral constants in the expression for $w$ can be determined from the following two conditions:

(1) The condition that the shear stress normal to the plane should be finite at $\zeta = 0$, and

(2) The boundary conditions on $w$ :

$$\left.\begin{aligned} &\zeta = 0 \quad ; \quad w_\zeta = 0 \\ &\zeta = 1 \quad ; \quad w = w_\zeta = 0 . \end{aligned}\right\} \tag{20}$$

The expression for $w$ of the second-order approximation is given by

$$w = \delta_o(1-\zeta^2)^2 + \frac{(1+\nu)xf(t)}{96}\delta\left[\left\{80 - \gamma(27+13\nu)\right\} - 12\left\{16 - \gamma(5+3\nu)\right\}\zeta^2\right.$$
$$\left. + 36\left\{4 - \gamma(1+\nu)\right\}\zeta^4 - 16(2-\nu\gamma)\zeta^6 + 3\gamma(1-\nu)\zeta^8\right] - \frac{(1+\nu)}{360}\frac{\delta(\delta^2-\delta_o^2)}{d^2}\left[(173-73\nu)\right.$$
$$- 6(77-37\nu)\zeta^2 + 90(5-3\nu)\zeta^4 - 20(11-9\nu)\zeta^6 + 75(1-\nu)\zeta^8 - 18(1-\nu)\zeta^{10}$$
$$\left. + 2(1-\nu)\zeta^{12}\right] + \frac{3(1+\nu)}{4}d\cdot xg(t)\gamma(1-\zeta^2)^2 \qquad (21)$$

where $x \equiv \alpha T_o(a^2/d^2)$ .

Finally, if the last condition is written as

$$\zeta = 0 \quad ; \quad w = \delta \qquad (22)$$

Eq.(21) leads to the following relation between the central deflection $\delta$ and the temperature rise:

$$\left(\frac{\delta}{d} - \frac{\delta_o}{d}\right) + \frac{(1+\nu)}{360}\left(\frac{\delta}{d}\right)\left(\frac{\delta^2}{d^2} - \frac{\delta_o^2}{d^2}\right)(173-73\nu) = \frac{3}{4}(1+\nu)xg(t)\gamma$$
$$+ \frac{(1+\nu)}{96}xf(t)\frac{\delta}{d}\left\{80 - \gamma(27+13\nu)\right\} \qquad (23)$$

## 4. DISCUSSION OF THE STABILITY CRITERION

Equation (23) is a cubic algebraic equation in $w$, which means that three equilibrium states can exist under the same temperature distribution. Which state of these three is stable and is actually occupied should be determined from the stability criterion.

As shown in the previous chapter, the nonlinear Euler equation can be solved comparatively easily by a progressive-approximation method, and in such cases, the stability criterion should rely on the method of "extremum" discussion in the variational theory. However, since it is very difficult to give a necessary and sufficient condition for stability, especially in the two-dimensional cases of nonlinear problems which include derivatives of more than the second order, an alternative approximation method for discussion of the stability criterion will be given.

The deviation of the total potential energy is given by

$$\Delta\pi = \delta\pi + \frac{1}{2!}\delta^2\pi + \ldots \tag{24}$$

and since $\delta\pi = 0$ from the stationary principle, the stability is determined by the sign of the second variation $\delta^2\pi$. If it is taken into account that $\delta^2\pi = \delta^2 F$ for the case of an isothermal change, the criterion is finally given as follows:

| | | |
|---|---|---|
| if | $\delta^2 F > 0$ | in stable equilibrium |
| if | $\delta^2 F = 0$ | at the limit of stability |
| if | $\delta^2 F < 0$ | in unstable equilibrium |

where $\delta^2 F$ is given by

$$\delta^2 F = \frac{2\pi E d a^2}{(1-\nu^2)} \int_0^1 \left[ \zeta \left( \frac{\delta u_\zeta}{a} + \frac{w_\zeta \delta w_\zeta}{a^2} \right)^2 + \zeta \frac{(\delta w_\zeta)^2}{a^2} \left\{ \frac{u_\zeta}{a} + \frac{1}{2a^2} (w_\zeta^2 - w_{o\zeta}^2) \right\} + \frac{(\delta u)^2}{a^2 \zeta} \right.$$

$$+ 2\nu \left\{ \frac{\delta u}{a} \left( \frac{\delta u_\zeta}{a} + \frac{w_\zeta \delta w_\zeta}{a^2} \right) + \frac{u}{2a} \frac{(\delta w_\zeta)^2}{a^2} + \frac{u}{2a} \frac{(\delta w_\zeta)^2}{a^2} \right\} + \frac{d^2}{12a^2} \left\{ \zeta \frac{(\delta w_{\zeta\zeta})^2}{a^2} \right.$$

$$\left. \left. + 2\nu \frac{\delta w_{\zeta\zeta}}{a} \frac{\delta w}{a} + \frac{1}{\zeta} \frac{(\delta w_\zeta)^2}{a^2} \right\} \right] d\zeta - \frac{2\pi E d a^2}{(1-\nu)} \int_0^1 \bar{T} \frac{(\delta w_\zeta)^2}{a^2} \zeta d\zeta \qquad (25)$$

The virtual displacements $\delta u$ and $\delta w$ are quite arbitrary as long as they are not inconsistent with the condition of $\delta \pi = 0$ and so the sign of $\delta^2 F$ should be discussed independently of $\delta u$ and $\delta w$. However, it is very difficult to obtain a necessary and sufficient condition for the discussion of the sign. Instead of using such an orthodox method, an assumption of appropriate functions of $\delta u$ and $\delta w$ makes it possible to evaluate approximately the sign of $\delta^2 F$ by means of Eq.(25), leaving some uncertainty because of the arbitrarinesses in the choice of the functions of $\delta u$ and $\delta w$. Therefore, an alternative method for the discussion of the stability criterion will be presented on the basis of the following logic.

$\delta^2 F$ can take various values depending on the assumed values of $\delta u$ and $\delta w$. Only the special modes of $\delta u$ and $\delta w$ which will lead to an extremum of $\delta^2 F$ are considered here. A variation of Eq.(25) with respect to $\delta u$ and $\delta w$ which is given by

$$\delta(\delta^2 F) = 0 \qquad (26)$$

leads to the following linear differential equations from which $\delta u$ and $\delta w$ can be determined:

$$\zeta \frac{d}{d\zeta} \left\{ \frac{1}{\zeta} \frac{d}{d\zeta} \left( \frac{\delta u}{a} \zeta \right) \right\} = - \left\{ (1-\nu) \frac{w_\zeta \delta w_\zeta}{a^2} + \zeta \frac{d}{d\zeta} \left( \frac{w_\zeta \delta w_\zeta}{a^2} \right) \right\} \qquad (27a)$$

$$\frac{d^2}{12a^2}\cdot\frac{d}{d\zeta}\left[\zeta\frac{d}{d\zeta}\left\{\frac{1}{\zeta}\frac{d}{d\zeta}\left(\frac{\delta w_\zeta}{a}\zeta\right)\right\}\right]=\frac{d}{d\zeta}\left[\zeta\frac{w_\zeta}{a}\left(\frac{\delta u_\zeta}{a}+\frac{w_\zeta\delta w_\zeta}{a^2}\right)\right.$$

$$\left.+\zeta\left\{\frac{u_\zeta}{a}+\frac{1}{2a^2}(w_\zeta^2-w_{o\zeta}^2)\right\}\frac{\delta w_\zeta}{a}+\nu\left(\frac{w_\zeta\delta u}{a^2}+\frac{u\delta w_\zeta}{a^2}\right)-\frac{\alpha(1+\nu)}{d}\bar{T}\frac{\delta w_\zeta}{a}\zeta\right] \tag{27b}$$

Now, the physical meaning of such special $\delta u$ and $\delta w$ values is to be examined. Eqs.(27) obtained from the condition of $\delta(\delta^2F)=0$ can also be derived by the following process: (1) the expansion of the equilibrium equations after inserting $u = u + \delta u$ and $w = w + \delta w$, and (2) the subtraction of the terms of the first order. This process leads to the same equation as Eq.(27). This means that the deformed state which deviated from an equilibrium state with special $\delta u$ and $\delta w$ values defined by Eq.(27) satisfies also the physical equilibrium conditions and these special virtual displacements are reasonable physically.

As a next step, it is necessary to check whether the extremum value of $\delta^2F$ is a maximum or a minimum as well as to judge the sign of $\delta^2F$. In order to determine this, $\delta^2(\delta^2F)$ should be further calculated by use of Eq.(25). Since $\delta^2F$ of Eq.(25) is a quadratic form in $\delta u$, $\delta w$ and their derivatives, $\delta^2(\delta^2F)$ has the same form with $\delta^2F$ except that $\delta u$ and $\delta w$ are replaced by $\delta(\delta u)$ and $\delta(\delta w)$. Consequently, it can be said that $\delta^2F$ and $\delta^2(\delta^2F)$ have the same sign under the deformed states in physical equilibrium. Finally, it can be concluded that

(1) if $\delta^2F$ is greater than zero, the $\delta u$ and $\delta w$ that satisfy the condition of $\delta(\delta^2F)=0$ give the minimum value of $\delta^2F$,

and (2) if $\delta^2F$ is smaller than zero, the $\delta u$ and $\delta w$ that satisfy the condition of $\delta(\delta^2F)=0$ give the maximum value of $\delta^2F$.

Consequently, it is sufficient to check only the sign of $\delta^2F$ of Eq.(25) by using $\delta u$ and $\delta w$ to be determined from Eqs.(27). However, there remains the problem of solving the linear homogeneous equations (27)

for $\delta u$ and $\delta w$, which is an eigenvalue problem. Taking into account that it is very difficult to solve these differential equations having variable coefficients, and that the physical deformations $u$ and $w$ obtained are only approximate ones, it seems to be enough to discuss the approximate solutions for $\delta u$ and $\delta w$. That is, first, a fundamental mode $\delta w$ is assumed so that it may give a minimum eigenvalue and then $\delta u$ is obtained from Eq.(27a) by using this $\delta w$. Finally, the sign of $\delta^2 F$ can be evaluated by use of such functions for $\delta u$ and $\delta w$.

In the present case, the following $\delta w$ which satisfies the condition of fixed edge will be considered:

$$\delta w = \Delta(1-\zeta^2)^2 \tag{28}$$

In the preceding paragraphs, a logic or process of discussing the stability criterion in nonlinear elastic problems was presented. As far as such a discussion of the stability criterion is concerned, the earlier research work by Marguerre on one-dimensional bars[1] should be appreciated. He discussed the stability limit on the basis that $\delta^2\pi$ is equal to zero at the stability limit and this zero value is the minimum value because the values cannot be negative. However, this logic seems to be a little ambiguous in a discussion of the stability criterion in the ordinary equilibrium states. The process itself in solving the equations derived from $\delta(\delta^2\pi) = 0$, which was suggested by him, is the same as that followed here, but the logic behind the process is different. The method described here can be considered a useful approximate method quite apart from the rigorous mathematical verification due to the variational discussion.

If there exist two stable equilibrium states under the same temperature distribution which have positive values of $\delta^2 F$, a further consideration of the energy levels in the two equilibrium states and during the buckling process will be necessary to determine which of the two is actually occupied.

## 5. NUMERICAL EXAMPLES

In the following numerical examples, $\nu$ is always taken as $1/3$.

(1) The case in which the temperature distribution through the thickness is specified as a linear one (see Figure 2).

It is assumed that the temperature distribution through the thickness is linear. Time effects are disregarded and the temperature on one surface is taken as twice that on the other surface, as shown in Figure 2.

(a) $\gamma = 0$, the case in which the temperature distribution over the surface is uniform.

A case is considered here in which the temperature on the upper surface is higher so that $f(t) = 1$ and $g(t) = 1/18$. The relationship between the nondimensional deflection ( $\delta/d$ ) and the temperature rise ( $x$ ) obtained by solving Eq.(23) is shown in Figure 3.

The relationship for the case with no initial deflection ( $\delta_o/d = 0$ ) is shown by the solid line. In this case, the plate does not deflect before the temperature reaches the following critical value $T_{ocr}$ :

$$T_{ocr} = \left(6/5(1+\nu)\alpha\right)(d^2/a^2) \tag{29}$$

and then it begins to deflect suddenly as seen in the Euler buckling.

On the other hand, when there exists any initial deflection, the plate deflects from the beginning of heating, as shown by the dotted line in Figure 3 for the case of $\delta_o/d = 0.1$. After the temperature reaches a certain high level, it becomes possible that the other two equilibrium states exist. However, the equilibrium curves at higher levels of the temperature are not connected with one another. This is different from the case of the snap-buckling of spherical shells or curved bars under external pressure[2,3]. Now, the stability of these three kinds of equilibrium states will be discussed.

From Eq.(21), the deflection $w$ for this case is given by

$$\frac{w}{d} = (1-\zeta^2)^2 \left[ \frac{\delta_o}{d} + \frac{2}{9} x(5-2\zeta^2)\left(\frac{\delta}{d}\right) - \frac{1}{405}\left(\frac{\delta}{d}\right)\left(\frac{\delta^2}{d^2} - \frac{\delta_o^2}{d^2}\right) \right.$$
$$\left. \times (223-136\zeta^2+45\zeta^4-14\zeta^6+2\zeta^8) \right]. \qquad (30)$$

After substitution of $w$ of Eq.(30) and $\delta w$ of Eq.(28) into the right-hand side of Eq.(27a), the progressive integration of it and the determination of the arbitrary integral constants from the following boundary conditions on $\delta u$ :

$$\zeta = 0,1 \quad ; \quad \delta u = 0 \qquad (31)$$

lead to the following expression for $\delta u$ :

$$\frac{\delta u}{d} = 4\left(\frac{\Delta}{d}\right)\left(\frac{d}{a}\right)\left[ \frac{\delta_o}{d}\frac{1}{9}(3\zeta-12\zeta^3+14\zeta^5-5\zeta^7) + x\left(\frac{\delta}{d}\right)\frac{1}{135}(51\zeta-240\zeta^3+350\zeta^5 \right.$$
$$- 200\zeta^7+39\zeta^9) - \frac{\delta}{d}\left(\frac{\delta^2}{d^2} - \frac{\delta_o^2}{d^2}\right)\frac{1}{8505}(1600\zeta-8148\zeta^3+13573\zeta^5-10500\zeta^7$$
$$\left. + 4641\zeta^9-1456\zeta^{11}+323\zeta^{13}-33\zeta^{15}) \right] \qquad (32)$$

Through substitution of $\delta u$ of Eq.(32) and $\delta w$ of Eq.(28) into Eq.(25), $\delta^2 F$ becomes

$$\frac{(1-\nu^2)}{2\pi E d^3}\left(\frac{a^2}{\Delta^2}\right)\delta^2 F = 0.8889(1-x) - 0.1862(y^2-y_o^2) + 0.8467y_o^2 + 1.7236x^2y^3$$
$$+ 0.4106y^2(y^2-y_o^2)^2 + 3.1605xyy_o - 1.485yy_o(y^2-y_o^2)$$
$$- 1.683/xy^2(y^2-y_o^2) \qquad (33)$$

where

$$y = \delta/d \quad , \quad y_o = \delta_o/d$$

The values of $\delta^2 F$ on the three kinds of equilibrium curves (I, II, III in Figure 3) are shown in Figure 4. It can be concluded that the equilibrium state (I) existing from the beginning of heating is stable because $\delta^2 F > 0$ and the downward deflected equilibrium states II and III are unstable and stable, respectively.

In order to check which of (I) and (III) is the reasonable equilibrium state during the heating process, the values of the total potential energy $\pi$ in these two states are calculated by means of Eq.(6) and are shown in Figure 5. A comparison of the values in the three equilibrium states shows that the energy level $\pi$ in (I) is lower than in (III) and $\pi$ in (II) is much higher than either one. Accordingly, the plate will deflect along curve (I) with an increase of temperature and it will be impossible for it to occupy the equilibrium state in (III) by snapping-through unless some external force is applied. Thus state (I) is the actual equilibrium state during the heating process.

Next, the deflection mode in state (I) will be considered. The deflection mode of the second order defined by Eq.(21) is shown in Figure 6 for the case $x = 2$ , as an example, for comparison with that of the first approximation. In this case of clamped edges, the deflection mode of the first approximation [Eq.(28)] satisfies the boundary condition of clamped edges and hence a large difference between them cannot be found; however, in the case of supported edges, the temperature distribution is influenced by the boundary conditions and so a careful assumption of the deflection mode of the first order will be required.

(b) $\gamma = 1$ , the case in which the temperature distribution over the surface is parabolic.

A case is considered here in which the temperature on the lower surface is higher, which can be expected to lead to snap buckling. The relationship between temperature and deflection is shown in Figure 7,

from which it can be seen that the plate starts to deflect from the beginning of heating owing to the pressure effect, even if there is no initial deflection. The effect of the temperature gradient for the case in which the temperature on the lower surface is higher corresponds to the action of a downward pressure, and so even the plate having a small upward initial deflection starts to deflect downwards. However, if the upward initial deflection is large, the plate continues to deflect upwards in spite of the downward-pressure effect because of the thermal expansion of the plane. The discussion of the stability criterion for this case leads to similar conclusions as described in (a), showing that the initial deformation state after the beginning of heating remains stable. The reason why the snap buckling phenomenon does not occur in spite of the external pressure effect in this case with initial deflection is the small pressure-effect due to a specified temperature distribution. In the following, a case will be considered in which the effect of temperature gradient is caused by instantaneous aerodynamic heating.

(2) The case in which the surface is exposed to aerodynamic heating instantaneously. ( $\gamma = 1$ ) .

In the heat-transfer due to aerodynamic heating which is expressed by $q = h(T_{aw}-T_w)$ , $h$ and $T_{aw} = (T_o)$ are considered to be kept constant after the instantaneous heating. The variations of $\bar{T}$ and $\tilde{T}$ can be given in the form of the following functions of time, respectively:

$$\left.\begin{aligned} f(t) &= 1 - 2\sum_{n=0}^{\infty} \frac{\sin^2 p_n \exp\left(-\dfrac{p_n^2 kt}{\rho C_p d^2}\right)}{p_n(p_n + \sin p_n \cos p_n)} \\ g(t) &= -\sum_{n=0}^{\infty} \frac{\sin p_n \{p_n \sin p_n + 2(\cos p_n - 1)\}}{p_n^2(p_n + \sin p_n \cos p_n)} \exp\left(-\frac{p_n^2 kt}{\rho C_p d^2}\right) \end{aligned}\right\} \qquad (34)$$

where $p_n$ denotes the roots satisfying the equation $p_n \tan p_n = hd/k$ (Biot Number). The functions $f(t)$ and $g(t)$ are shown in Figure 8 for the two kinds of Biot number.

A cooling process will also be considered in which $T_{aw}$ is instantaneously reduced to the initial temperature after being heated uniformly due to a sustained heating.

The variation of deflection ( $\delta/d$ ) with the lapse of time ( $kt/\rho C_p d^2$ ) is shown in Figure 9 for the case of $x = 3$ where there is no initial deflection and the Biot number is assumed to be kept constant. The deflections during the heating and cooling process are shown by the fine and heavy lines, respectively. During the heating process, the plate will deflect along the fine solid line marked (I). During the cooling process after settling down to an equilibrium state by heating, the deflection of the plate decreases with the lapse of time along the heavy solid line marked (I') and then snaps through suddenly to the downward-deflected state marked (III'). With the lapse of time and a decrease in the temperature, the plate will return gradually to its original flat state.

The deformation process above described can be verified from the following discussion of the stability criterion: The deflection $w$ under the temperature distribution given by Eq.(34) is

$$\frac{w}{d} = \left\{\frac{\delta_o}{d} + xg(t)\right\}(1-\zeta^2)^2 + \frac{xf(t)}{108}\frac{\delta}{d}(73-180\zeta^2+140\zeta^4-40\zeta^6+3\zeta^8)$$

$$- \frac{1}{405}\frac{\delta}{d}\frac{\delta^2-\delta_o^2}{d^2}(223-582\zeta^2+540\zeta^4-240\zeta^6+75\zeta^8-18\zeta^{10}+2\zeta^{12}) \qquad (35)$$

because of Eq.(21). The virtual displacement $\delta u$ obtained from Eq.(21) is

$$\frac{\delta u}{d} = \frac{4}{9}\left(\frac{d}{a}\right)\left(\frac{\Delta}{d}\right)\left[\left\{\frac{\delta_o}{d}\, xg(t)\right\}(3-36\zeta^2+70\zeta^4-35\zeta^6) + \frac{1}{180}\, xf(t)\, \frac{\delta}{d}\,(373-5400\zeta^2\right.$$

$$+ 13650\zeta^4-11900\zeta^6+3861\zeta^8-352\zeta^{10}) - \frac{1}{945}\,\frac{\delta}{d}\,\frac{\delta^2-\delta_o^2}{d^2}\,(1600-24444\zeta^2$$

$$\left. + 67865\zeta^4-73500\zeta^6+41769\zeta^8-16016\zeta^{10}+4199\zeta^{12}-495\zeta^{14})\right] \qquad (36)$$

If the expressions of Eqs.(19), (35) and (36) are used, $\delta^2 F$ can be obtained as follows:

$$\frac{81}{32}\,\frac{1-\nu^2}{\pi E d^3}\left(\frac{a^2}{\Delta^2}\right)\delta^2 F = 4.5 - 2.7xf(t) - 0.9428(y^2-y_o^2) + 4.286\left(y_o+xg(t)\right)^2$$

$$+ 3.086xg(t)\left\{xg(t)+2y_o\right\} + 3.005x^2f^2(t)y^2$$

$$+ 2.074y^2(y^2-y_o^2)^2 + 9.594xf(t)y\left(y_o+xg(t)\right)$$

$$- 7.801y(y^2-y_o^2)\left(y_o+xg(t)\right) - 5.174xf(t)y^2(y^2-y_o^2) \qquad (37)$$

The variations in the values of $\delta^2 F$ on each of the equilibrium curves in Figure 9 are shown in Figure 10. From these curves, it can be easily seen that the equilibrium states (I) and (III) are stable, but (II) is unstable during the heating process, and the plate deflects continuously along curve (I), without exhibiting snap-through buckling. On the other hand, the equilibrium states (I') during the cooling process become unstable and the plate snaps through to the stable equilibrium state of (III'). It should be noted that a snap buckling may be induced during the cooling process after heating, even if there is no initial deflection.

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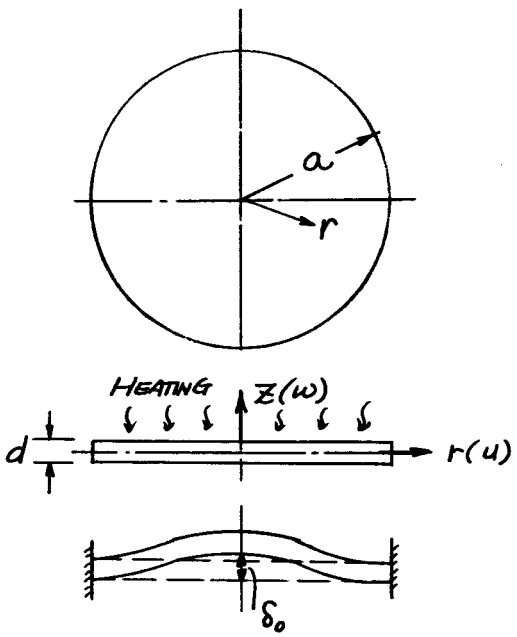



FIG. 1

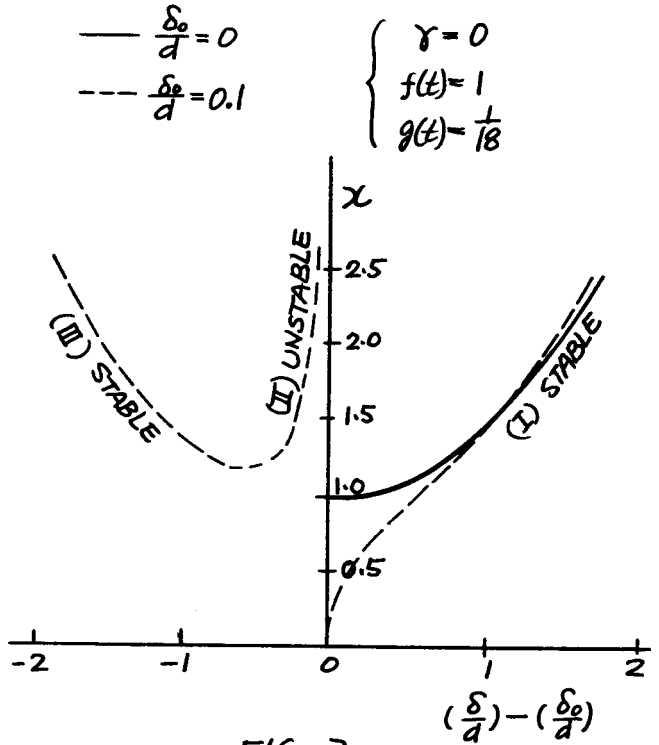



FIG. 3

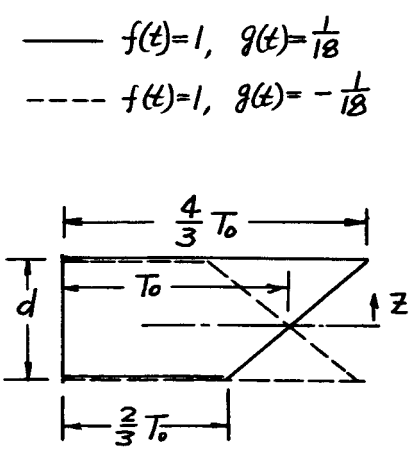



FIG. 2

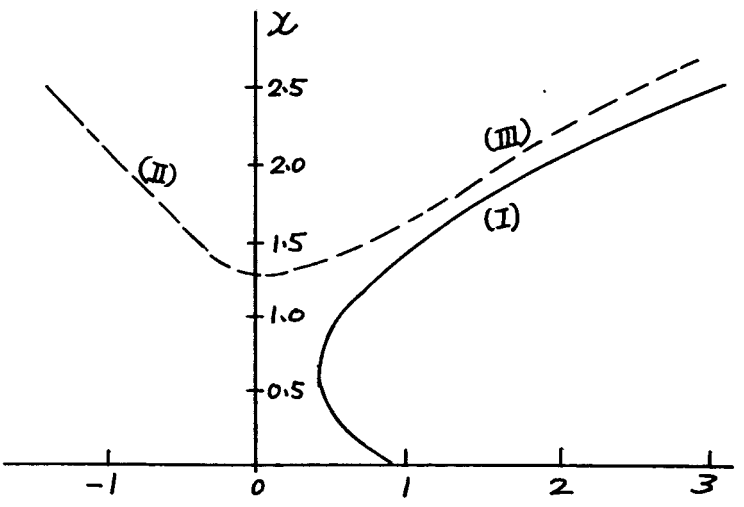
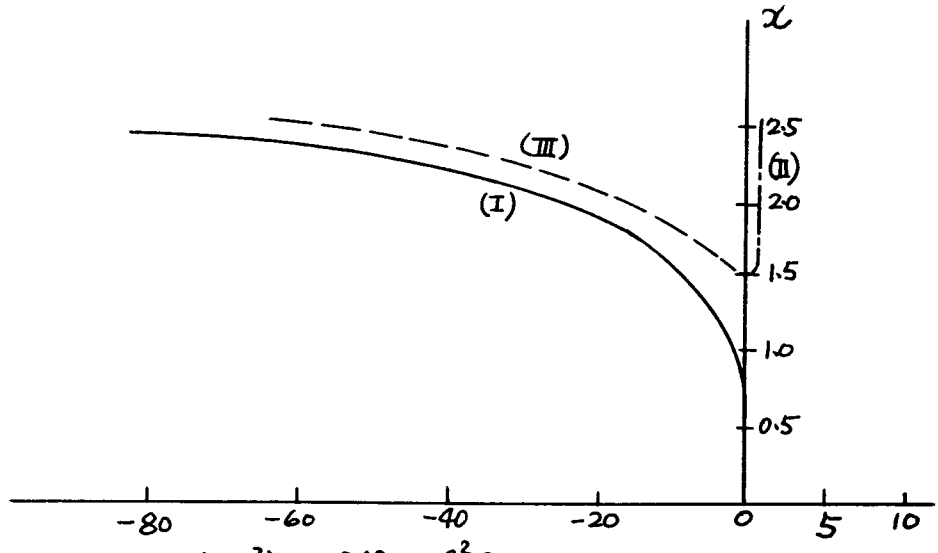



FIG. 4




$\frac{(1-\nu^2)}{\pi E d a^2} \times \frac{243}{16} \left(\frac{a^2}{d^2}\right)^2 \pi$

FIG. 5

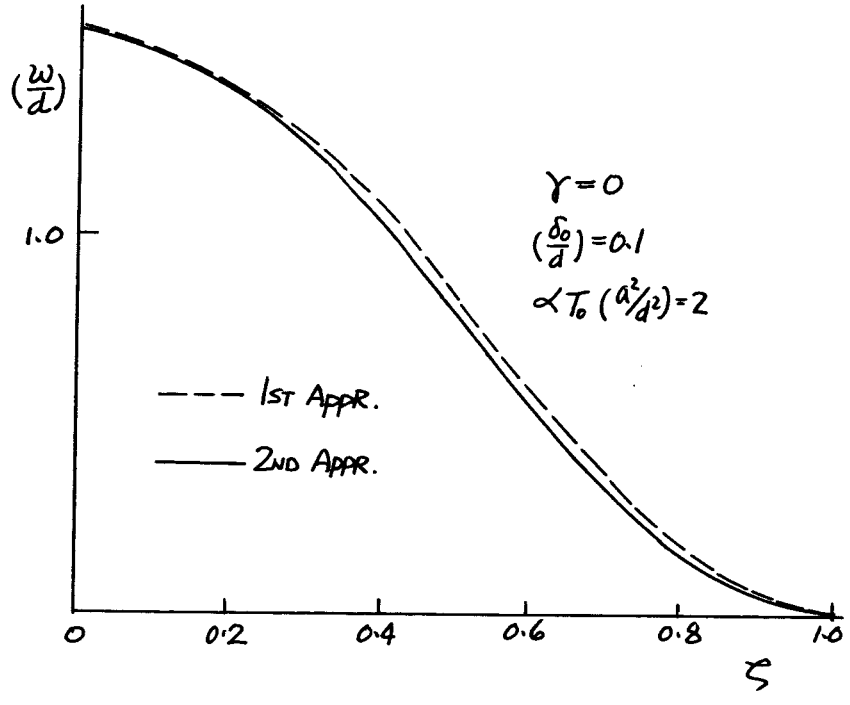



FIG. 6

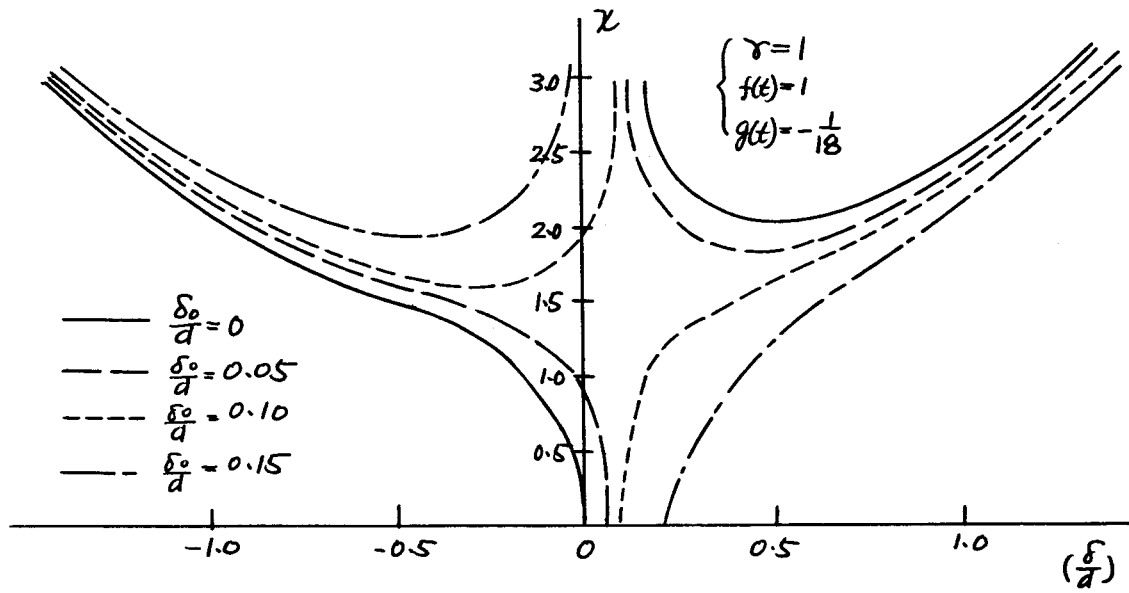

χ
γ=1
f(t)=1
g(t)=-1/18
3.0
2.5
2.0
1.5
1.0
0.5
δ₀/d = 0
δ₀/d = 0.05
δ₀/d = 0.10
δ₀/d = 0.15
-1.0
-0.5
0
0.5
1.0
(δ/d)

FIG. 7

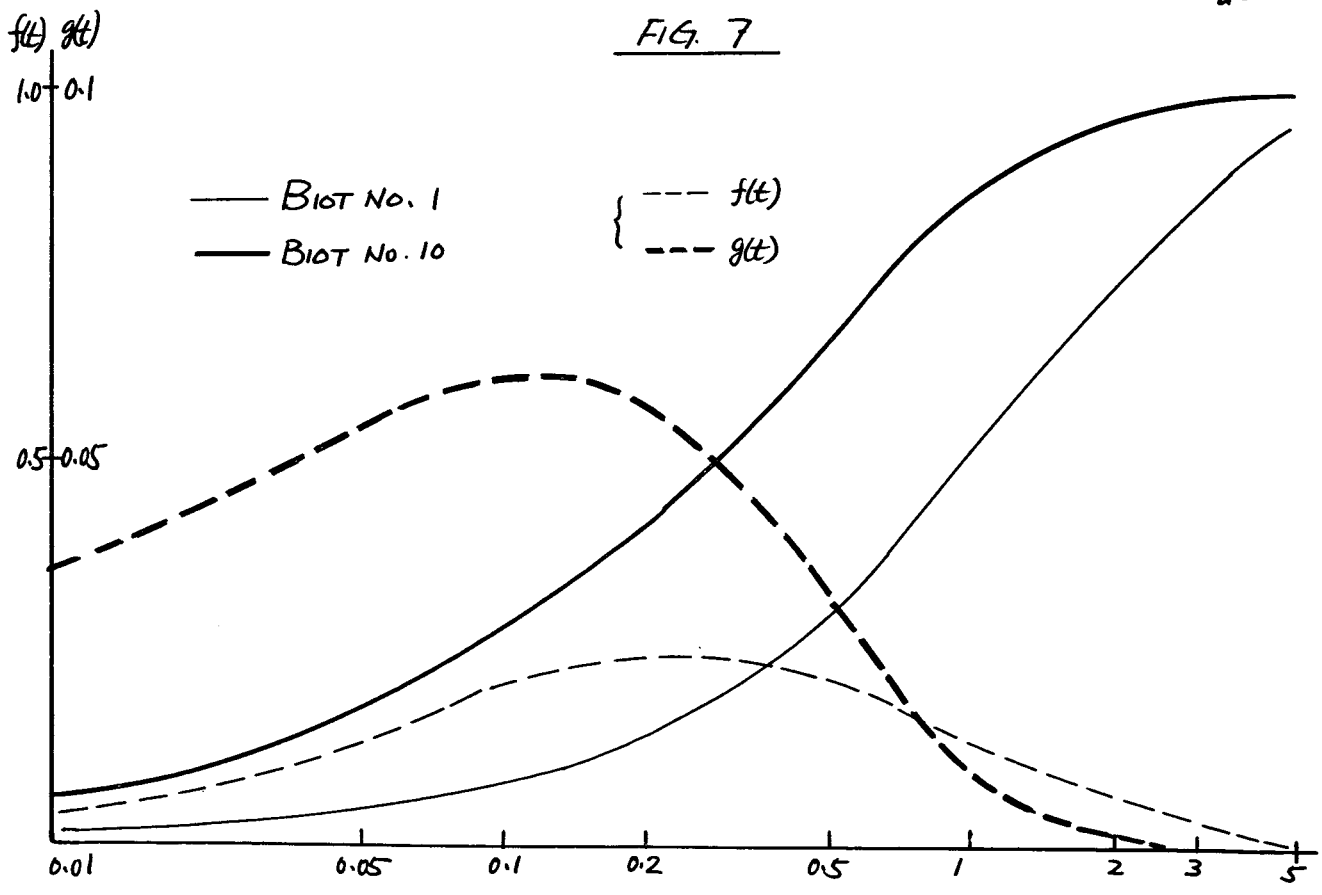

f(t) g(t)
1.0
0.1
0.5
0.05
BIOT NO. 1
BIOT NO. 10
f(t)
g(t)
0.01
0.05
0.1
0.2
0.5
1
2
3
5
Kt/ρCpd²

FIG. 8

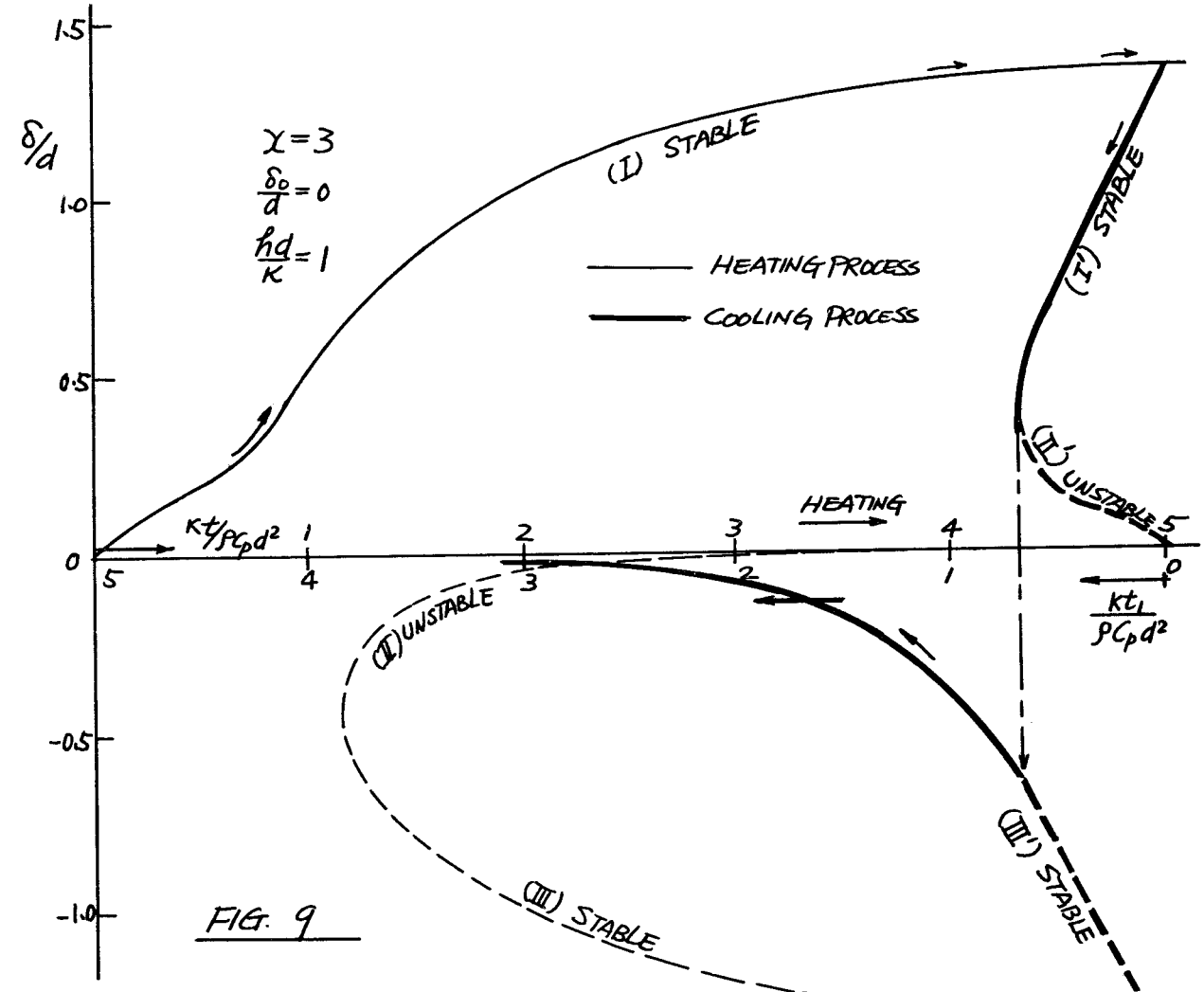

δ/d
χ=3
δ₀/d=0
hd/κ=1
(I) STABLE
HEATING PROCESS
COOLING PROCESS
(I') STABLE
(II') UNSTABLE
HEATING
Kt/ρC_pd²
kt_L/ρC_pd²
(II) UNSTABLE
(III) STABLE
(III') STABLE

FIG. 9

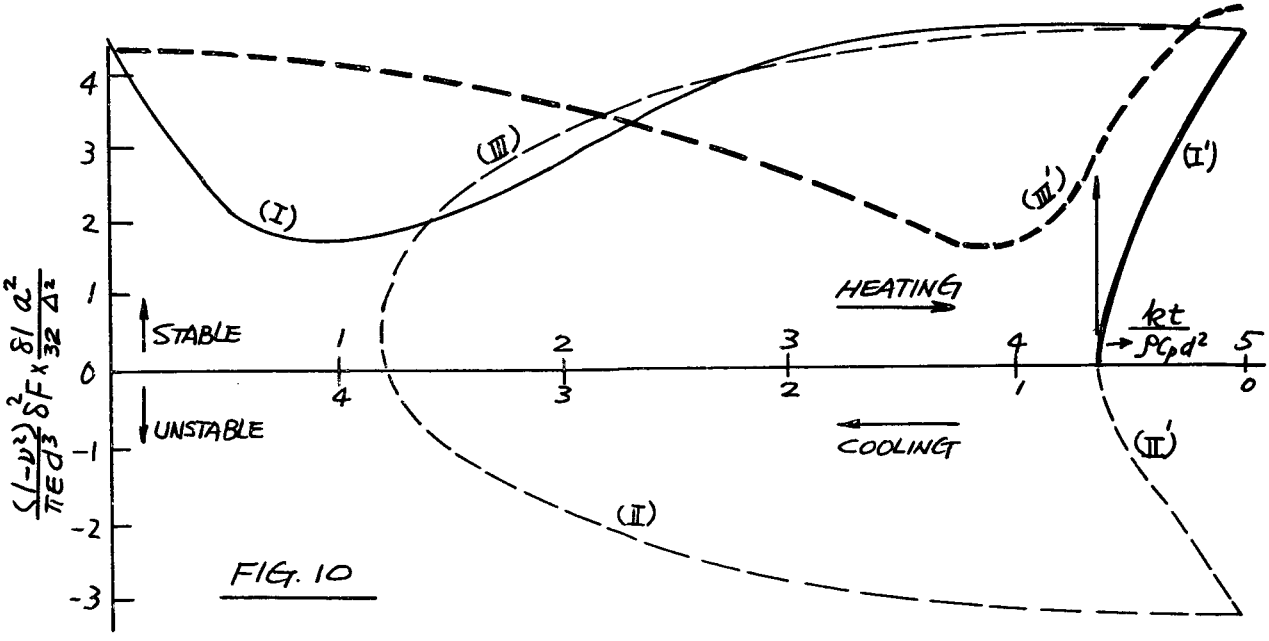

(I)
(II)
(III)
(I')
(II')
(III')
HEATING
COOLING
STABLE
UNSTABLE
kt/ρC_pd²

FIG. 10